

Parallel Partition

Input: An array $A[q : r]$ of distinct elements, and an element x from $A[q : r]$.

Output: Rearrange the elements of $A[q : r]$, and return an index $k \in [q, r]$, such that all elements in $A[q : k - 1]$ are smaller than x , all elements in $A[k + 1 : r]$ are larger than x , and $A[k] = x$.

```
Par-Partition (  $A[ q : r ]$ ,  $x$  )  
1.  $n \leftarrow r - q + 1$   
2. if  $n = 1$  then return  $q$   
3. array  $B[ 0 : n - 1 ]$ ,  $lt[ 0 : n - 1 ]$ ,  $gt[ 0 : n - 1 ]$   
4. parallel for  $i \leftarrow 0$  to  $n - 1$  do  
5.    $B[ i ] \leftarrow A[ q + i ]$   
6.   if  $B[ i ] < x$  then  $lt[ i ] \leftarrow 1$  else  $lt[ i ] \leftarrow 0$   
7.   if  $B[ i ] > x$  then  $gt[ i ] \leftarrow 1$  else  $gt[ i ] \leftarrow 0$   
8.  $lt[ 0 : n - 1 ] \leftarrow \textit{Par-Prefix-Sum}$  (  $lt[ 0 : n - 1 ]$ ,  $+$  )  
9.  $gt[ 0 : n - 1 ] \leftarrow \textit{Par-Prefix-Sum}$  (  $gt[ 0 : n - 1 ]$ ,  $+$  )  
10.  $k \leftarrow q + lt[ n - 1 ]$ ,  $A[ k ] \leftarrow x$   
11. parallel for  $i \leftarrow 0$  to  $n - 1$  do  
12.   if  $B[ i ] < x$  then  $A[ q + lt[ i ] - 1 ] \leftarrow B[ i ]$   
13.   else if  $B[ i ] > x$  then  $A[ k + gt[ i ] ] \leftarrow B[ i ]$   
14. return  $k$ 
```

Parallel Partition

A:

9	5	7	11	1	3	8	14	4	21
---	---	---	----	---	---	---	----	---	----

 $x = 8$

Parallel Partition

A:

9	5	7	11	1	3	8	14	4	21
---	---	---	----	---	---	---	----	---	----

 $x = 8$

B:

0	1	2	3	4	5	6	7	8	9
9	5	7	11	1	3	8	14	4	21

lt:

0	1	2	3	4	5	6	7	8	9
0	1	1	0	1	1	0	0	1	0

gt:

0	1	2	3	4	5	6	7	8	9
1	0	0	1	0	0	0	1	0	1

Parallel Partition

A:

9	5	7	11	1	3	8	14	4	21
---	---	---	----	---	---	---	----	---	----

x = 8

B:

0	1	2	3	4	5	6	7	8	9
9	5	7	11	1	3	8	14	4	21

lt:

0	1	2	3	4	5	6	7	8	9
0	1	1	0	1	1	0	0	1	0

lt:

0	1	2	2	3	4	4	4	5	5
---	---	---	---	---	---	---	---	---	---

prefix sum

gt:

0	1	2	3	4	5	6	7	8	9
1	0	0	1	0	0	0	1	0	1

gt:

1	1	1	2	2	2	2	3	3	4
---	---	---	---	---	---	---	---	---	---

prefix sum

Parallel Partition

A:

9	5	7	11	1	3	8	14	4	21
---	---	---	----	---	---	---	----	---	----

x = 8

	0	1	2	3	4	5	6	7	8	9
B:	9	5	7	11	1	3	8	14	4	21

	0	1	2	3	4	5	6	7	8	9
<i>It:</i>	0	1	1	0	1	1	0	0	1	0

It:	0	1	2	2	3	4	4	4	5	5
------------	---	---	---	---	---	---	---	---	---	---

prefix sum

gt:

0	1	2	3	4	5	6	7	8	9
1	0	0	1	0	0	0	1	0	1

gt:

1	1	1	2	2	2	2	3	3	4
---	---	---	---	---	---	---	---	---	---

prefix sum

[illegible]

Parallel Partition

A:

9	5	7	11	1	3	8	14	4	21
---	---	---	----	---	---	---	----	---	----

x = 8

B:

0	1	2	3	4	5	6	7	8	9
9	5	7	11	1	3	8	14	4	21

lt:

0	1	2	3	4	5	6	7	8	9
0	1	1	0	1	1	0	0	1	0

lt:

0	1	2	2	3	4	4	4	5	5
---	---	---	---	---	---	---	---	---	---

prefix sum

k = 5

gt:

0	1	2	3	4	5	6	7	8	9
1	0	0	1	0	0	0	1	0	1

gt:

1	1	1	2	2	2	2	3	3	4
---	---	---	---	---	---	---	---	---	---

prefix sum

A:

0	1	2	3	4	5	6	7	8	9
5	7	1	3	4	8				

Parallel Partition

A:

9	5	7	11	1	3	8	14	4	21
---	---	---	----	---	---	---	----	---	----

x = 8

B:

0	1	2	3	4	5	6	7	8	9
9	5	7	11	1	3	8	14	4	21

lt:

0	1	2	3	4	5	6	7	8	9
0	1	1	0	1	1	0	0	1	0

lt:

0	1	2	2	3	4	4	4	5	5
---	---	---	---	---	---	---	---	---	---

prefix sum

k = 5

gt:

0	1	2	3	4	5	6	7	8	9
1	0	0	1	0	0	0	1	0	1

gt:

1	1	1	2	2	2	2	3	3	4
---	---	---	---	---	---	---	---	---	---

prefix sum

A:

0	1	2	3	4	5	6	7	8	9
5	7	1	3	4	8	9	11	14	21

Parallel Partition

A:

9	5	7	11	1	3	8	14	4	21
---	---	---	----	---	---	---	----	---	----

x = 8

B:

0	1	2	3	4	5	6	7	8	9
9	5	7	11	1	3	8	14	4	21

lt:

0	1	2	3	4	5	6	7	8	9
0	1	1	0	1	1	0	0	1	0

gt:

0	1	2	3	4	5	6	7	8	9
1	0	0	1	0	0	0	1	0	1

lt:

0	1	2	2	3	4	4	4	5	5
---	---	---	---	---	---	---	---	---	---

gt:

1	1	1	2	2	2	2	3	3	4
---	---	---	---	---	---	---	---	---	---

prefix sum

k = 5

prefix sum

A:

0	1	2	3	4	5	6	7	8	9
5	7	1	3	4	8	9	11	14	21

Parallel Partition: Analysis

Par-Partition ($A[q : r], x$)

1. $n \leftarrow r - q + 1$
2. *if* $n = 1$ *then return* q
3. *array* $B[0 : n - 1], lt[0 : n - 1], gt[0 : n - 1]$
4. *parallel for* $i \leftarrow 0$ *to* $n - 1$ *do*
5. $B[i] \leftarrow A[q + i]$
6. *if* $B[i] < x$ *then* $lt[i] \leftarrow 1$ *else* $lt[i] \leftarrow 0$
7. *if* $B[i] > x$ *then* $gt[i] \leftarrow 1$ *else* $gt[i] \leftarrow 0$
8. $lt[0 : n - 1] \leftarrow \text{Par-Prefix-Sum} (lt[0 : n - 1], +)$
9. $gt[0 : n - 1] \leftarrow \text{Par-Prefix-Sum} (gt[0 : n - 1], +)$
10. $k \leftarrow q + lt[n - 1], A[k] \leftarrow x$
11. *parallel for* $i \leftarrow 0$ *to* $n - 1$ *do*
12. *if* $B[i] < x$ *then* $A[q + lt[i] - 1] \leftarrow B[i]$
13. *else if* $B[i] > x$ *then* $A[k + gt[i]] \leftarrow B[i]$
14. *return* k

Work:

$$\begin{aligned} T_1(n) &= \Theta(n) && [\text{lines } 1 - 7] \\ &+ \Theta(n) && [\text{lines } 8 - 9] \\ &+ \Theta(n) && [\text{lines } 10 - 14] \\ &= \Theta(n) \end{aligned}$$

Span:

Assuming $\log n$ depth for *parallel for* loops:

$$\begin{aligned} T_\infty(n) &= \Theta(\log n) && [\text{lines } 1 - 7] \\ &+ \Theta(\log^2 n) && [\text{lines } 8 - 9] \\ &+ \Theta(\log n) && [\text{lines } 10 - 14] \\ &= \Theta(\log^2 n) \end{aligned}$$

Parallelism: $\frac{T_1(n)}{T_\infty(n)} = \Theta\left(\frac{n}{\log^2 n}\right)$

Randomized Parallel QuickSort

Input: An array $A[q : r]$ of distinct elements.

Output: Elements of $A[q : r]$ sorted in increasing order of value.

Par-Randomized-QuickSort ($A[q : r]$)

1. $n \leftarrow r - q + 1$
2. *if* $n \leq 30$ *then*
3. sort $A[q : r]$ using any sorting algorithm
4. *else*
5. select a random element x from $A[q : r]$
6. $k \leftarrow \text{Par-Partition} (A[q : r], x)$
7. *spawn* *Par-Randomized-QuickSort* ($A[q : k - 1]$)
8. *Par-Randomized-QuickSort* ($A[k + 1 : r]$)
9. *sync*

Randomized Parallel QuickSort: Analysis

```
Par-Randomized-QuickSort (  $A[q : r]$  )  
1.  $n \leftarrow r - q + 1$   
2. if  $n \leq 30$  then  
3.   sort  $A[q : r]$  using any sorting algorithm  
4. else  
5.   select a random element  $x$  from  $A[q : r]$   
6.    $k \leftarrow \text{Par-Partition} ( A[q : r], x )$   
7.   spawn Par-Randomized-QuickSort (  $A[q : k - 1]$  )  
8.   Par-Randomized-QuickSort (  $A[k + 1 : r]$  )  
9.   sync
```

Lines 1—6 take $\Theta(\log^2 n)$ parallel time and perform $\Theta(n)$ work.

Also the recursive spawns in lines 7—8 work on disjoint parts of $A[q : r]$. So the upper bounds on the parallel time and the total work in each level of recursion are $\Theta(\log^2 n)$ and $\Theta(n)$, respectively.

Hence, if D is the *recursion depth* of the algorithm, then

$$T_1(n) = O(nD) \text{ and } T_\infty(n) = O(D \log^2 n)$$

Randomized Parallel QuickSort: Analysis

```
Par-Randomized-QuickSort (  $A[q : r]$  )  
1.  $n \leftarrow r - q + 1$   
2. if  $n \leq 30$  then  
3.   sort  $A[q : r]$  using any sorting algorithm  
4. else  
5.   select a random element  $x$  from  $A[q : r]$   
6.    $k \leftarrow \text{Par-Partition} ( A[q : r], x )$   
7.   spawn Par-Randomized-QuickSort (  $A[q : k - 1]$  )  
8.   Par-Randomized-QuickSort (  $A[k + 1 : r]$  )  
9.   sync
```

We will show that w.h.p. recursion depth, $D = O(\log n)$.

Hence, with high probability,

$$T_1(n) = O(n \log n) \text{ and } T_\infty(n) = O(\log^3 n)$$

Randomized Parallel QuickSort: Analysis

Approach: We will show the following

1. For any specific element v , the sizes of the partitions containing v in any two consecutive levels of recursion decrease by a constant factor with a certain probability.
2. With probability $1 - O\left(\frac{1}{n^7}\right)$, the partition containing v will be of size 30 or less after $O(\log n)$ levels of recursion.
3. With probability $1 - O\left(\frac{1}{n^6}\right)$, the partition containing every element will be of size 30 or less after $O(\log n)$ levels of recursion.

Randomized Parallel QuickSort: Analysis

Lemma 1: Let v be an arbitrary element of the original input array A of size $n = n_0$, and let n_j be the size of the partition containing v after partitioning at recursion depth $j \geq 1$. Then for any $j \geq 0$,

$$\Pr \left[n_{j+1} \geq \frac{7}{8} n_j \right] \leq \frac{1}{4}.$$

Proof: Suppose at recursion depth $j + 1 \geq 1$ element x was chosen as the pivot element.

One of the new partitions will have at least $\frac{7}{8} n_j$ elements provided x is among the smallest or largest $\frac{1}{8} n_j$ elements in the old partition.

The probability that x is among the smallest or largest $\frac{1}{8} n_j$ elements in the old partition is clearly $\leq \frac{1}{8} + \frac{1}{8} = \frac{1}{4}$.

Randomized Parallel QuickSort: Analysis

Lemma 2: In $20 \log n$ levels of recursion, the probability that an element goes through $20 \log n - \log(n/30)$ *unsuccessful partitioning steps* (i.e., partitioning steps with $n_{j+1} < (7/8)n_j$) is $O(1/n^7)$.

[all logarithms are to the base $8/7$]

Proof: The events consisting of the partitioning steps being successful can be modeled as *Bernoulli trials*.

Let X be a random variable denoting the number of unsuccessful partitioning steps among the $20 \log n$ steps. Then

$$\Pr[X > 20 \log n - \log(n/30)] \leq \Pr[X > 19 \log n]$$

$$\leq \sum_{j > 19 \log n} \binom{20 \log n}{j} \left(\frac{1}{4}\right)^j \left(\frac{3}{4}\right)^{20 \log n - j}$$

Randomized Parallel QuickSort: Analysis

Lemma 2: In $20 \log n$ levels of recursion, the probability that an element goes through $20 \log n - \log(n/30)$ *unsuccessful partitioning steps* (i.e., partitioning steps with $n_{j+1} < (7/8)n_j$) is $O(1/n^7)$.

[all logarithms are to the base $8/7$]

Proof: $\Pr[X > 20 \log n - \log(n/30)] \leq \Pr[X > 19 \log n]$

$$\begin{aligned} &\leq \sum_{j > 19 \log n} \binom{20 \log n}{j} \left(\frac{1}{4}\right)^j \left(\frac{3}{4}\right)^{20 \log n - j} \\ &\leq \sum_{j > 19 \log n} \left(\frac{20e \log n}{j}\right)^j \left(\frac{1}{4}\right)^j = \sum_{j > 19 \log n} \left(\frac{5e \log n}{j}\right)^j \\ &\leq \sum_{j > 19 \log n} \left(\frac{5e \log n}{19 \log n}\right)^j = \sum_{j > 19 \log n} \left(\frac{5e}{19}\right)^j = O\left(\frac{1}{n^7}\right) \end{aligned}$$

Randomized Parallel QuickSort: Analysis

Theorem 1: The recursion depth of *Par-Randomized-Quicksort* is $\leq 20\log_{8/7}n$ with probability $1 - O(1/n^6)$.

Proof: The probability that one or more elements of A go through $20\log_{8/7}n - \log_{8/7}(n/30)$ unsuccessful partitioning steps is

$$O\left(n \times \frac{1}{n^7}\right) = O\left(\frac{1}{n^6}\right) \text{ [using Lemma 2]}$$

Hence, the probability that at least $\log_{8/7}(n/30)$ of the $20\log_{8/7}n$ partitioning steps is successful for all elements is $1 - O(1/n^6)$.

After $t = \log_{8/7}(n/30)$ successful partitioning steps involving an element, the element belongs to a partition of size $\left(\frac{7}{8}\right)^t n = 30$.

Hence, *Par-Randomized-Quicksort* terminates in $\leq 20\log_{8/7}n$ levels of recursion with probability $1 - O(1/n^6)$.

Parallel Selection

Input: A subarray $A[q : r]$ of an array $A[1 : n]$ of n distinct elements, and a positive integer $k \in [1, r - q + 1]$.

Output: An element x of $A[q : r]$ such that $\text{rank}(x, A[q : r]) = k$.

Par-Selection ($A[q : r], n, k$)

1. $n' \leftarrow r - q + 1$
2. *if* $n' \leq n / \log n$ *then*
3. sort $A[q : r]$ using a *parallel sorting algorithm* and *return* $A[q + k - 1]$
4. *else*
5. partition $A[q : r]$ into blocks B_i 's each containing $\log n$ consecutive elements
6. *parallel for* $i \leftarrow 1$ *to* $\lceil n' / \log n \rceil$ *do*
7. $M[i] \leftarrow$ median of B_i using a *sequential selection algorithm*
8. find the median m of $M[1 : \lceil n' / \log n \rceil]$ using a *parallel sorting algorithm*
9. $t \leftarrow \text{Par-Partition}$ ($A[q : r], m$)
10. *if* $k = t - q + 1$ *then return* $A[t]$
11. *else if* $k < t - q + 1$ *then return* *Par-Selection* ($A[q : t - 1], n, k$)
12. *else return* *Par-Selection* ($A[t + 1 : r], n, k - t + q - 1$)

Parallel Selection

Lemma 3: In *Par-Selection* (lines 11—12)

$$|A[q:t-1]| \leq \frac{3n'}{4} \text{ and } |A[t+1:r]| \leq \frac{3n'}{4}.$$

Proof: It suffices to show that $\frac{n'}{4} \leq \text{rank}(m, A[q:r]) \leq \frac{3n'}{4}$.

Since m is the median of $M[i]$'s, it is larger than one half of the $M[i]$'s. But each $M[i]$ is larger than $\frac{\log n}{2}$ elements in B_i .

$$\text{Hence, } \text{rank}(m, A[q:r]) \geq \frac{n'}{2 \log n} \times \frac{\log n}{2} = \frac{n'}{4}.$$

Similarly, one can show that $\text{rank}(m, A[q:r]) \leq \frac{3n'}{4}$.

Parallel Selection

Lemma 4: In *Par-Selection* $n' \leq \frac{n}{\log n}$ after at most $\log_{4/3} \log n$ levels of recursion.

Proof: It follows from Lemma 3 that $n' \leq \left(\frac{3}{4}\right)^k n$ after k levels of recursion.

Hence, for reaching $n' \leq \frac{n}{\log n}$, we need

$$\frac{n}{\log n} \geq \left(\frac{3}{4}\right)^k n \Rightarrow k \leq \log_{4/3} \log n.$$

Deterministic Parallel Selection

Par-Selection ($A[q : r]$, n , k)

1. $n' \leftarrow r - q + 1$
2. *if* $n' \leq n / \log n$ *then*
3. sort $A[q : r]$ using a *parallel sorting algorithm* and *return* $A[q + k - 1]$
4. *else*
5. partition $A[q : r]$ into blocks B_i 's each containing $\log n$ consecutive elements
6. *parallel for* $i \leftarrow 1$ *to* $\lceil n' / \log n \rceil$ *do*
7. $M[i] \leftarrow$ median of B_i using a *sequential selection algorithm*
8. find the median m of $M[1 : \lceil n' / \log n \rceil]$ using a *parallel sorting algorithm*
9. $t \leftarrow$ *Par-Partition* ($A[q : r]$, m)
10. *if* $k = t - q + 1$ *then return* $A[t]$
11. *else if* $k < t - q + 1$ *then return* *Par-Selection* ($A[q : t - 1]$, n , k)
12. *else return* *Par-Selection* ($A[t + 1 : r]$, n , $k - t + q - 1$)

Step 7: Use a linear time (worst-case) sequential selection algorithm (see Section 9.3 of “Introduction to Algorithms”, 3rd Ed. by Cormen et al.).

Steps 3 and 8: Use the parallel mergesort with parallel merge (see Lecture 8) that runs in $O(\log^3 n)$ parallel time and performs $O(n \log n)$ work in the worst case.

Deterministic Parallel Selection

Par-Selection ($A[q : r]$, n , k)

1. $n' \leftarrow r - q + 1$
2. *if* $n' \leq n / \log n$ *then*
3. sort $A[q : r]$ using a *parallel sorting algorithm* and *return* $A[q + k - 1]$
4. *else*
5. partition $A[q : r]$ into blocks B_i 's each containing $\log n$ consecutive elements
6. *parallel for* $i \leftarrow 1$ *to* $\lceil n' / \log n \rceil$ *do*
7. $M[i] \leftarrow$ median of B_i using a *sequential selection algorithm*
8. find the median m of $M[1 : \lceil n' / \log n \rceil]$ using a *parallel sorting algorithm*
9. $t \leftarrow$ *Par-Partition* ($A[q : r]$, m)
10. *if* $k = t - q + 1$ *then return* $A[t]$
11. *else if* $k < t - q + 1$ *then*
 return Par-Selection ($A[q : t - 1]$, n , k)
12. *else return Par-Selection* ($A[t + 1 : r]$, n , $k - t + q - 1$)

Last Level of Recursion

Work: $O\left(\frac{n}{\log n} \log\left(\frac{n}{\log n}\right)\right) = O(n)$

Span: $O\left(\log^3\left(\frac{n}{\log n}\right)\right) = O(\log^3 n)$

Any Other Recursion Level (except last)

Work: $O\left(\frac{n'}{\log n} \times \log n\right)$ [lines 5 – 7]

+ $O\left(\frac{n'}{\log n} \log\left(\frac{n'}{\log n}\right)\right)$ [line 8]

+ $O(n')$ [line 9]

= $O(n')$

Span: $O\left(\log\left(\frac{n'}{\log n}\right) + \log n\right)$ [lines 5 – 7]

+ $O\left(\log^3\left(\frac{n'}{\log n}\right)\right)$ [line 8]

+ $O(\log^2 n')$ [line 9]

= $O(\log^3 n)$

Deterministic Parallel Selection

Par-Selection ($A[q : r]$, n , k)

1. $n' \leftarrow r - q + 1$
2. *if* $n' \leq n / \log n$ *then*
3. sort $A[q : r]$ using a *parallel sorting algorithm* and *return* $A[q + k - 1]$
4. *else*
5. partition $A[q : r]$ into blocks B_i 's each containing $\log n$ consecutive elements
6. *parallel for* $i \leftarrow 1$ *to* $\lceil n' / \log n \rceil$ *do*
7. $M[i] \leftarrow$ median of B_i using a *sequential selection algorithm*
8. find the median m of $M[1 : \lceil n' / \log n \rceil]$ using a *parallel sorting algorithm*
9. $t \leftarrow$ *Par-Partition* ($A[q : r]$, m)
10. *if* $k = t - q + 1$ *then return* $A[t]$
11. *else if* $k < t - q + 1$ *then*
 return *Par-Selection* ($A[q : t - 1]$, n , k)
12. *else return* *Par-Selection* ($A[t + 1 : r]$,
 n , $k - t + q - 1$)

Overall

Work:

$$T_1(n) = O\left(n + \sum_{i=0}^{\log_{4/3} \log n} \left(\frac{3}{4}\right)^i n\right) \\ = O(n)$$

Span:

$$T_\infty(n) = O\left((\log_{4/3} \log n) \log^3 n\right) \\ = O(\log^3 n \log \log n)$$